Correlation Between Eigenvalue Spectra and Dynamics of Neural Networks

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This letter presents a study of the correlation between the eigenvalue spectra of synaptic matrices and the dynamical properties of asymmetric neural networks with associative memories. For this type of neural network, it was found that there are essentially two different dynamical phases: the chaos phase, with almost all trajectories converging to a single chaotic attractor, and the memory phase, with almost all trajectories being attracted toward fixed-point attractors acting as memories. We found that if a neural network is designed in the chaos phase, the eigenvalue spectrum of its synaptic matrix behaves like that of a random matrix (i.e., all eigenvalues lie uniformly distributed within a circle in the complex plan), and if it is designed in the memory phase, the eigenvalue spectrum will split into two parts: one part corresponds to a random background, the other part equal in number to the memory attractors. The mechanism for these phenomena is discussed in this letter.

The statistical properties of the eigenvalues of random matrices have been of great interest and a major focus in mathematics and physics (Metha, 2006). One of the well-known achievements is the Wigner semicircle law (Verbaarschot & Zirnbauer, 1984), which states that for N-dimensional real symmetric matrices whose elements are drawn from a gaussian distribution with zero mean and variance $1/N$, the eigenvalues distribute over a finite interval $(-2, 2)$ with average density $\rho(\lambda) = (2\pi)^{-1}(4 - \lambda^2)^{1/2}$. Such investigation has been subsequently generalized to the asymmetric cases. It is found that the eigenvalues of asymmetric random matrix uniformly

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Neural Computation **21, 2931–2941** (2009)

lie within one circle in the complex plane (Girko, 1984), and the circle changes to an ellipse if partial symmetry is considered (Sommers, Crisanti, Sompolinsky, & Stein, 1988). Since the pioneering work of Wigner in the 1950s, the eigenvalue spectra of random matrices have been proven to be valuable for understanding the dynamical behaviors of systems far from equilibrium (Brody et al., 1981). Particularly, they may provide useful clues about the behavior of feedback neural networks (Crisanti & Sompolinsky, 1987; Rajan & Abbott, 2006).

A feedback neural network is composed of N artificial neurons connected to each other through synaptic matrix $J_{ij}(i, j = 1, \ldots, N)$ and evolves according to given dynamical principles. The most commonly used dynamical principle, which is also adopted in this letter, is expressed as

$$S_i(t+1) = sgn(h_i(t)), \quad h_i(t) = \sum_{j=1}^{N} J_{ij} S_j(t), \quad i = 1, \ldots, N, \tag{1}$$

where $S_i(t) \in \{+1, -1\}$ represents the state of ith neuron at time t and $h_i(t)$ is the local field. The feedback neural network belongs to an important type of dynamical system. In the view of statistical mechanics, they can be regarded as disordered magnetic systems (Schuster, 2001); therefore, the dynamic and thermodynamic properties have attracted extensive attention (Amit, Gutfreund, & Sompolinsky, 1985). In view of practical application, they have the capability of content-addressable memory (Hopfield, 1984), promising great potential in associative memory (Kohonen, 1984) and pattern recognition (Zhao, 2004; Jin & Zhao, 2005).

Relevant research on the correlation between eigenvalue spectrum of synaptic matrix and dynamical behavior of feedback neural network, to the best of our knowledge, can be traced back to the work of Crisanti and Sompolinsky (1987). They found in Hopfield neural networks that the maximal eigenvalue of synaptic matrix would define the critical temperature at which the spin-glass state emerged. Subsequent studies show that for a symmetric neural network with associative memories, the minimal eigenvalue defines a threshold value of neuron gain above which the system converging to a fixed-point attractor or not is not guaranteed (Waugh, Marcus, & Westervelt, 1990). Recently Rajan and Abbott (2006) studied the eigenvalue spectra of random matrices constructed by excitatory and inhibitory columns that satisfy different probability distributions. It was found that eigenvalues of those matrices would also lie within a circle if a certain balance condition is satisfied (Troyer & Miller, 1997) but would not distribute uniformly unless the variances of different distributions were equal. They suggest that the appearance of eigenvalues at the edge of a circle implies the existence of slow-oscillating and long-lasting modes.

Though much work has been done, there is still a lack of direct evidence to attest that the eigenvalue spectrum of a synaptic matrix can reflect the dynamical behavior of a feedback neural network, just like that for the billiard system (Baker, Schubert, & Stifter, 1998). In this letter, we study the eigenvalue spectra of synaptic matrices of asymmetric neural networks with associative memories. The intrinsic correlation between the eigenvalue spectrum of a synaptic matrix and the dynamical behavior of the neural network is our focus. Such an investigation was first proposed by one of the authors (Zhou, 2002).

For feedback neural networks with associative memories, one basic problem is to find a synaptic matrix J_{ij}, which guarantees that a given set of p memories $\{\xi_i^\mu\}$ ($i = 1, \ldots, N, \mu = 1, \ldots, p$) is a fixed-point attractor of the system. To be a fixed-point attractor, the memory should satisfy $\{\xi_i^\mu = sgn(\sum_{j=1}^N J_{ij}\xi_j^\mu), i = 1, \ldots, N\}$ as its dynamical principle, equation 1, or equivalently,

$$h_i^\mu \geq \kappa, \quad h_i^\mu = \xi_i^\mu \left(\sum_{j=1}^N J_{ij}\xi_j^\mu \right), \quad i = 1, \ldots, N, \tag{2}$$

with $\kappa > 0$. The existence probability of such a matrix has been studied carefully as the functions of the parameter κ, the storage ratio $\alpha = p/N$, the memory correlativity, and the symmetry of synaptic matrix (Gardner, 1988; Gardner, Gutfreund, & Yekutieli, 1989). In spite of this, the explicit solution of J_{ij} is practically obtained by the so-called learning rules (Hagan, Demuth, & Beale, 2002).

Two types of learning rules are considered in this letter. The first one is the generalized perceptron rule (Gardner, 1988). This is a local iterative rule by which the randomly initialized synaptic matrices are continuously updated until all of the memories satisfy equation 2. Its convergence follows from a generalization of the perceptron convergence theorem (Minsky & Papert, 1969). The second one is the Monte Carlo adaptation (MCA) rule (Zhao, 2004). This is also an iterative improvement rule. Unlike the generalized perceptron rule, it is a global algorithm. When this rule is used, each update of the synaptic matrix is the optimal choice for all of the memories. The MCA rule is more effective and flexible in designing neural networks with a discrete synaptic matrix (Gutfreund & Stein, 1990). Both the generalized perceptron rule and the MCA rule can be applied to design asymmetric neural networks with controllable κ.

The parameter κ is significant (Gardner, 1988). Recent studies have shown that the dynamical behaviors of asymmetric neural networks designed by the MCA rule are quite different in different ranges of κ (Zhao, 2004). Generally, this parameter can be divided into three intervals, respectively, corresponding to three different dynamical phases: (1) a chaos phase, in which the neural network has a chaotic attractor coexisting with memories; (2) a memory phase, in which the memories are the only stable attractors; and (3) a mixture phase, in which metastable fixed-point attractors appear to coexist with the memories. The discovery of these dynamical phases provides an important motivation for our research. In

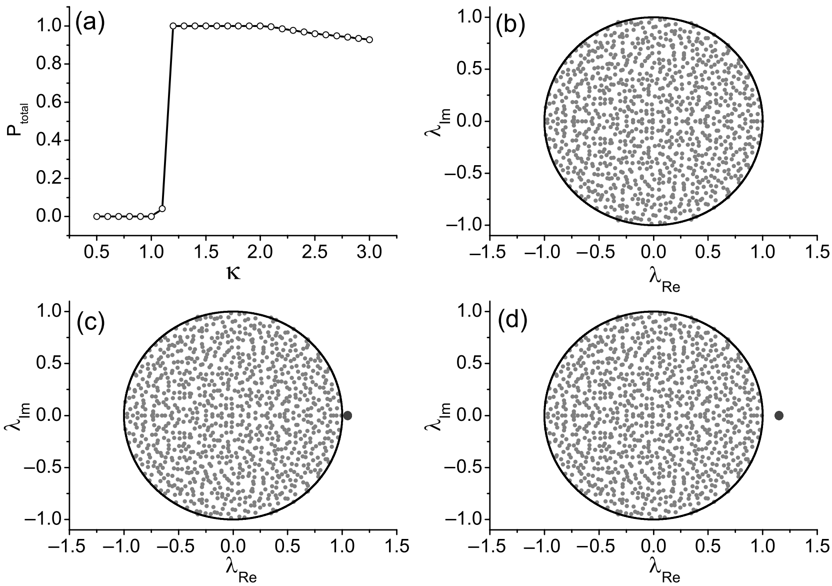

Figure 1: Results of neural networks with $h_i^\mu = \kappa$. (a) P_{total} versus κ. (b) Eigenvalue spectrum with $\kappa = 0.95$. (c) Eigenvalue spectrum with $\kappa = 1.05$. (d) Eigenvalue spectrum with $\kappa = 1.15$.

fact, they are a fundamental behavior of asymmetric neural networks with associative memories. To illustrate this statement, we study the dynamical behaviors of neural networks designed by the generalized perceptron rule.

In the beginning, the extreme case of condition 2, $h_i^\mu = \kappa$ $(i = 1, \ldots, N)$, for all memories, is considered. To obtain an expectant synaptic matrix, the relaxation version of the generalized perceptron rule (Gardner et al., 1989) is employed,

$$J_{ij}^{(m+1)} = J_{ij}^{(m)} + \frac{1}{N}\left(\kappa - h_i^\mu\right)\xi_i^\mu \xi_j^\mu, \quad i, j = 1, \ldots, N, \tag{3}$$

where m indicates the number of iterations.

First we confirm the dynamical phases in neural networks designed by equation 3. For this purpose, we compute the percentage of random initial states converged to those memories under the evolution of equation 1, denoted by P_{total}, with different κ as in Zhao (2004). The results are shown in Figure 1a for the case of $p = 10$ and $N = 1000$. Each P_{total} is obtained by averaging over 10 independent systems. Each system is prepared in the following way. Initialize the synaptic matrix, for convenience of analysis, whose elements are restricted by

$$\langle J_{ij} \rangle = 0, \quad \langle J_{ij}^2 \rangle = \frac{1}{N}, \tag{4}$$

where $\langle \cdots \rangle$ represents an average of the distribution; randomly select p system states to be memories; iteratively apply equation 3 while keeping equation 4 until $|h_i^\mu - \kappa| < 10^{-6}$ is achieved for all the memories. For each prepared system, we check 10,000 random initial states to get P_{total}. As usual, state $\{-\xi_i^\mu\}$ is equivalent to $\{\xi_i^\mu\}$ for the symmetry of equation 1. The same parameters are also used in the following numerical calculation.

Figure 1a indicates that the parameter κ can be divided into three ranges with two turning points, $\kappa_1 \approx 1.0$ and $\kappa_2 \approx 2.0$. In detail, in the range of $\kappa \leq \kappa_1$, $P_{total} = 0$; in the range of $\kappa_1 < \kappa \leq \kappa_2$, $P_{total} = 1$; beyond κ_2, the value of P_{total} decreases with the increase of κ. Further studies show that the dynamical behaviors of neural networks designed in those different ranges are similar to those reported in Zhao (2004). In other words, these three ranges correspond to the chaos phase, the memory phase, and the mixture phase, respectively. Here, we emphasize that $P_{total} = 0$ means that almost all the random initial states are attracted to the chaotic attractor, and the memories are locally stable; $P_{total} = 1$ means that almost all the random initial states are attracted to the memories, which are globally stable.

We then compute the eigenvalue spectra of synaptic matrices designed by equation 3 with different κ as proposed above. Some typical results are shown in Figure 1. It can be seen that the eigenvalue spectrum in the chaos phase is essentially different from that in the memory phase. In the chaos phase, the eigenvalue spectrum behaves much like that of the asymmetric random matrix (Girko, 1984): all eigenvalues uniformly lie within a circle of complex plane as shown in Figure 1b. In the memory phase, the eigenvalue spectrum splits into two parts, as shown in Figure 1d. The first part still lies inside the circle, called the *noise part*; the second part lies outside, in the *information part*. Further studies show that the eigenvalue spectrum splits if and only if $\kappa > \kappa_1$ and that the eigenvalues belonging to the information part are equal in number to the memories. The mechanism of these phenomenon is presented in the following.

The synaptic matrix J_{ij} with $h_i^\mu = \kappa$ can be written as

$$J_{ij} = \kappa J_{ij}^d + J_{ij}^r. \tag{5}$$

As long as p remains finite, the randomly selected memories are effectively orthogonal or linearly independent in the thermodynamic limit ($N \to \infty$). For this case, the matrix J_{ij}^d can be obtained by the Hebbian rule $J_{ij}^d = \frac{1}{N}\sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu$; therefore, it is symmetric with $\langle J_{ij}^d \rangle = 0$ and $\langle (J_{ij}^d)^2 \rangle = \frac{p}{N^2}$ (McEliced, Posner, Rodemick, & Venkatesh, 1987). Suppose that the matrix

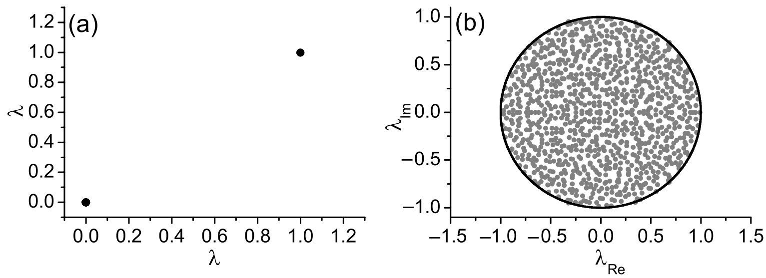

Figure 2: Eigenvalue spectra of (a) matrix J^d_{ij} and (b) matrix J^r_{ij} with $h^\mu_i = 1.05$.

J_{ij} is asymmetric and satisfies equation 4; then matrix J^r_{ij} is also asymmetric, and one may easily find that $\langle J^r_{ij}\rangle = 0$ and $\langle (J^r_{ij})^2\rangle = \frac{1}{N}(1-\alpha\kappa^2)$ because $\langle J^d_{ij} J^r_{ij}\rangle = 0$ as $N \to \infty$.

One can derive $\xi^\mu_i(\sum_{j=1}^N J^d_{ij}\xi^\mu_j) = 1$ and $\xi^\mu_i(\sum_{j=1}^N J^r_{ij}\xi^\mu_j) = 0$ from the Hebb rule and equation 5 for all of the memories $\{\xi^\mu_i\}(\mu = 1, \ldots, p)$ with $i = 1, \ldots, N$. This means that all memories are common eigenvectors of matrix J^d_{ij} and matrix J^r_{ij} with eigenvalues $\lambda^d_\mu = 1$ and $\lambda^r_\mu = 0$, respectively. Furthermore, the remaining eigenvectors of matrix J^r_{ij}, denoted by $\{\zeta^l_i\}(l = p+1, \ldots, N)$, are also the eigenvectors of matrix J^d_{ij} with eigenvalues $\lambda^d_l = 0$ because $\sum_{i=1}^N \xi^\mu_i \zeta^l_i = 0$. Therefore, matrix J^d_{ij} has only two degenerate eigenvalues: 0 and 1 (Waugh et al., 1990). To get a better visual impression, they are plotted on the $\lambda \sim \lambda$ plane, as shown in Figure 2a. On the other hand, according to knowledge of the eigenvalue spectra of random asymmetric matrices (Girko, 1984; Rajan & Abbott, 2006), one may learn that all the eigenvalues of matrix J^r_{ij} uniformly lie within a circle of the complex plane as shown in Figure 2b, centered at the origin with radius $R = (1-\alpha\kappa^2)^{\frac{1}{2}}$. It implies that the radius of the circle decreases with the increase of k for a fixed storage ratio α.

Based on the preceding analysis, we know that the common eigenvectors of J^r_{ij} and J^d_{ij} are also the eigenvectors of $J_{ij}(= \kappa J^d_{ij} + J^r_{ij})$ with eigenvalues $\lambda_i = \kappa\lambda^d_i + \lambda^r_i$. That is, the eigenvalue spectrum of matrix J_{ij} is a linear superposition of the eigenvalue spectra of matrix J^d_{ij} and matrix J^r_{ij}. When parameter κ satisfies

$$\kappa > (1-\alpha\kappa^2)^{\frac{1}{2}}, \tag{6}$$

the eigenvalue spectrum of matrix J_{ij} splits into two parts because there are p eigenvalues that lie outside the circle with radius $R = (1-\alpha\kappa^2)^{\frac{1}{2}}$. One may obtain $\kappa > 0.99$ in the case of $\alpha = 0.01$, which is consistent with our numerical results $\kappa > \kappa_1 \approx 1.0$. In fact, for finite p, it is true that $\kappa > 1.0$ when $N \to \infty$.

Since synaptic matrix J_{ij} can be expressed by the linear superposition of symmetric matrix J_{ij}^d and random asymmetric matrix J_{ij}^r, the local field acting on the ith neuron can be expressed as

$$h_i = \kappa \sum_{j=1}^{N} J_{ij}^d S_j + \sum_{j=1}^{N} J_{ij}^r S_j. \tag{7}$$

It implies that the state of ith neuron can be regarded as the result of competition between two mechanisms. Considering only the effect of matrix J_{ij}^d, one may define

$$H = -\frac{\kappa}{2} \sum_{i,j=1}^{N} J_{ij}^d S_i S_j, \tag{8}$$

which has been proven to be monotonic decreasing and behaves as the Hamiltonian of the neural networks described by equation 1 (Hopfield, 1984). The effect of matrix J_{ij}^r is as follows. Since $\langle J_{ij}^r \rangle = 0$ and $\langle (J_{ij}^r)^2 \rangle = \frac{1}{N}(1 - \alpha\kappa^2)$ while $|S_j| = 1$, the second term on the right-hand side of equation is a sum of N independent and identically distributed random variables with zero mean and variance $\frac{1}{N}(1 - \alpha\kappa^2)$. The central limit theorem tells us that all $(\sum_{j=1}^{N} J_{ij}^r S_j)$ satisfy the gaussian distribution with zero mean and variance $(1 - \alpha\kappa^2)$ when $N \to \infty$. As a consequence, the asymmetric feedback neural network described by equation 1 with $h_i^\mu = \kappa$ is equivalent to the kinetic Ising model (Glauber, 1963) with Hamiltonian (see equation 8) and effective temperature $\beta = 1/(1 - \alpha\kappa^2)^{\frac{1}{2}}$.

It is known that under the kinetic Ising model, the state of a system approaches the equilibrium Gibbs-Boltzmann distribution,

$$P(S) = \frac{\exp(-\beta H(S))}{Z} \tag{9}$$

even when the initial condition is away from equilibrium. Thus, the dynamical behaviors of the neural networks mentioned above can be analyzed by the equilibrium statistical mechanics after a sufficiently long time from the random initial states. The corresponding partition function is

$$Z = Tr_s \exp(-\beta H) = Tr_s \exp\left(\frac{\tilde{\beta}}{2N} \sum_{\mu=1}^{p} \left(\sum_{i=1}^{N} \xi_i^\mu S_i \right)^2 \right), \tag{10}$$

with $\tilde{\beta} = \kappa/(1 - \alpha\kappa^2)^{\frac{1}{2}}$. Here, Tr_s is the sum over all possible state S. Such a system has been studied intensively by applying mean-field theory (Amit et al., 1985). It is found that the system changes from the paramagnetic to the ferromagnetic phase when $\tilde{\beta} > 1$, that is, $\kappa > (1 + \alpha)^{-1/2}$.

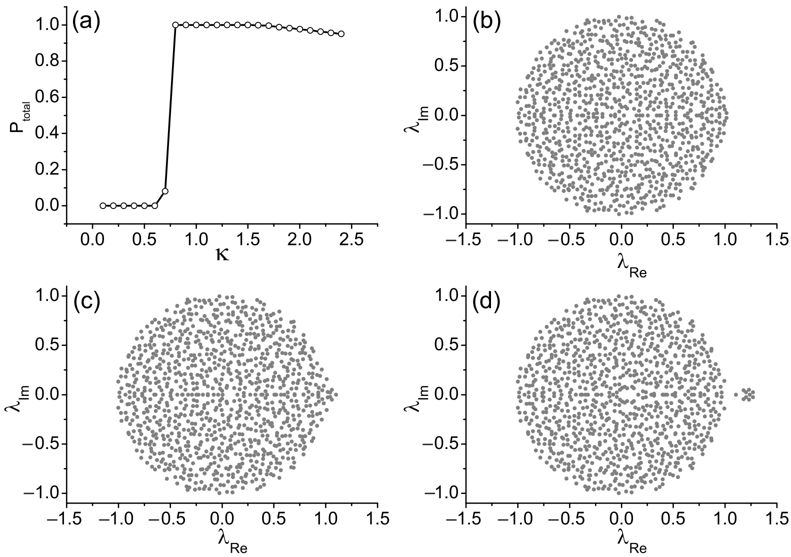

Figure 3: Results of neural networks with $h_i^\mu \geq \kappa$. (a) P_{total} versus κ. (b) Eigenvalue spectrum with $\kappa = 0.55$. (c) Eigenvalue spectrum with $\kappa = 0.65$. (d) Eigenvalue spectrum with $\kappa = 0.75$.

In the paramagnetic phase, the overlaps $m^\mu = \frac{1}{N}\sum_{i=1}^{N} S_i \xi_i^\mu$ between the equilibrium state and the memories are 0. In the ferromagnetic phase, the overlaps are given by $m^\mu = 1$ when $1 < \tilde{\beta} < \frac{1}{0.461}$, that is, $(1+\alpha)^{-1/2} < \kappa < (0.461^2 + \alpha)^{-1/2}$, while the metastable state appears for $\tilde{\beta} > \frac{1}{0.461}$. Notice that the condition for phase transition is the same as equation 6, under which the eigenvalue spectrum of synaptic matrix splits into two parts. That is, the paramagnetic phase intrinsically corresponds to the chaos phase, while the ferromagnetic phase corresponds to the memory phase and mixture phase, in agreement with our numerical experiments.

By decomposing the synaptic matrix, we have shown that the asymmetric neural network described by equation 1 with $h_i^\mu = \kappa$ is equivalent to the kinetic Ising model described by equation 8 with effective temperature $\beta = 1/(1-\alpha\kappa^2)^{\frac{1}{2}}$ in the case of $N \to \infty$ and finite p. As a consequence, the eigenvalue spectrum of the synaptic matrix behaves much like that of an asymmetric random matrix if the dynamical behavior of the neural network is irregular, but it splits into two parts if the dynamical behavior of the neural network is regular. Actually, this correspondence is quite universal. It is found not only in the case of $h_i^\mu = \kappa$ but also in the case of $h_i^\mu \geq \kappa$ with different types of synaptic matrices, as shown in the following.

Figure 3 shows the results of neural networks designed by the original generalized perceptron rule, $J_{ij}^{(m+1)} = J_{ij}^{(m)} + \frac{1}{N}\varepsilon_i^\mu \xi_i^\mu \xi_j^\mu$, where $\varepsilon_i^\mu =$

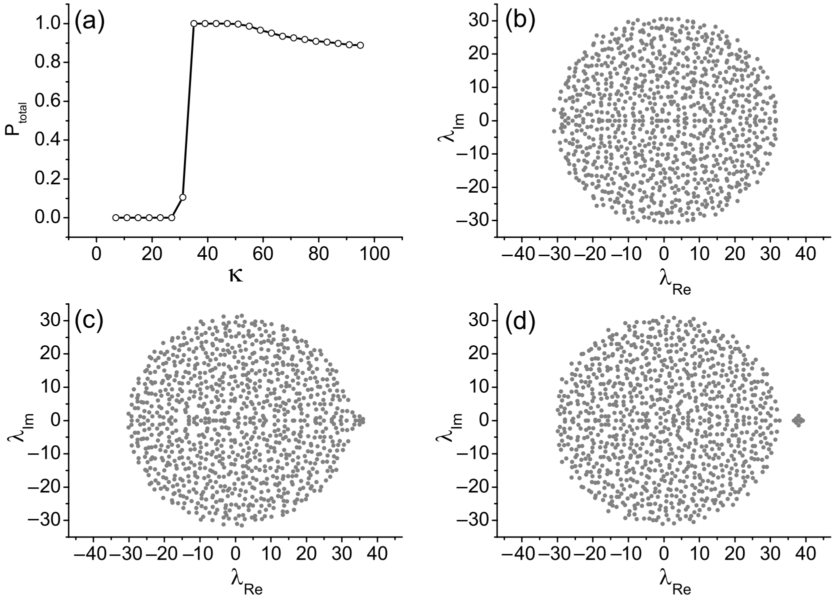

Figure 4: Results of neural networks designed by the MCA rule. (a) P_{total} versus κ. (b) Eigenvalue spectrum with $\kappa = 24$. (c) Eigenvalue spectrum with $\kappa = 28$. (d) Eigenvalue spectrum with $\kappa = 32$.

$\theta[\kappa(\sum_{j=1}^{N} J_{ij}^2)^{\frac{1}{2}} - h_i^\mu]$ and $\theta(x)$ is the step function (Gardner, 1988). The synaptic matrices satisfy equation 2: $h_i^\mu \geq \kappa$. Here condition 4 still holds. Comparing Figure 3 with Figure 1, one can notice that the main results obtained in the case of $h_i^\mu \geq \kappa$ are qualitatively the same as those obtained in the case of $h_i^\mu = \kappa$. In detail, the parameter κ can also be delimited into three intervals, as shown in Figure 3a, corresponding to the chaos phase, the memory phase, and the mixture phase, respectively. The eigenvalue of synaptic matrices uniformly lies within a circle of a complex plan in the chaos phase, as shown in Figure 3b. The eigenvalue spectrum of the synaptic matrix splits into the "noise part" and the "information part" in the memory phase, as shown in Figure 3d. The main difference is $\kappa_1 \approx 0.6$ in the case of $h_i^\mu \geq \kappa$, while $\kappa_1 \approx 1.0$ in the case of $h_i^\mu = \kappa$. We emphasize that eigenvalues belonging to the information part are no longer equal to each other in the case of $h_i^\mu \geq \kappa$.

Figure 4 shows the results of neural networks designed by the MCA rule. Here, we keep $|J_{ij}| = 1$ as in Zhao (2004); therefore, the synaptic matrix satisfies $\langle J_{ij} \rangle = 0$ and $\langle J_{ij}^2 \rangle = 1$ in the thermodynamic limit. Condition 4 is satisfied for this case. There is no qualitative difference between Figure 4 and Figure 3, except that $\kappa_1 \approx 26$ and the radius of the noise part is $R = (N - \alpha\kappa^2)^{1/2}$ in Figure 4.

In conclusion, we have studied the intrinsic correlation between the eigenvalue spectra of synaptic matrices and dynamical behavior of asymmetric neural networks with associative memories. It has been found that the eigenvalue spectra behave like those of random matrices if the network's behavior is chaotic after a sufficiently long time from the random initial states, but they split into two parts if the network's behavior is regular. Moreover, this phenomenon depends on neither the distribution of local field h_i^μ caused by the memories nor the value range of the synaptic matrix J_{ij}; it is quite universal. The mechanism of such phenomena has been carefully studied. We discovered that the asymmetric neural networks with $h_i^\mu = \kappa$ are equivalent to the kinetic Ising model at an appropriate temperature. This equivalence results in direct correspondence between the eigenvalue spectrum and the dynamical behavior observed in asymmetric neural networks. It would be interesting to extend the results of this work to a more general case.

Acknowledgments

We thank Nicholas McGuire for helpful discussions and suggestions and H. Schuster for useful discussions and leading us to some useful references. We also thank Zhang Yong, Hu Bin, and Xiao Junjun for their constructive discussion. This work is supported in part by the National Natural Science Foundation of China under grant no. 10475067 and the Major State Research Development 973 Project of Nonlinear Science in China.

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Received December 17, 2007; accepted February 17, 2009.